

Noise effects in the quantum search algorithm from the computational complexity point of view

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We analyse the resilience of the quantum search algorithm in the presence of quantum noise modelled as trace preserving completely positive maps. We study the influence of noise on computational complexity of the quantum search algorithm. We show that only for small amounts of noise the quantum search algorithm is still more efficient than any classical algorithm.

Keywords: quantum algorithms, quantum noise, algorithms complexity

I. INTRODUCTION

It is often said that the strength of quantum computation lies in the phenomena of quantum superposition and quantum entanglement. These features of quantum computation allow to perform the computation on all possible inputs that fit to the quantum register. One of the greatest achievements in the theory of quantum algorithms is quantum search algorithm introduced by Grover. The detailed description of this algorithm can be found in [1], [2], [3], [4].

Any physical implementation of a quantum computer will be error-prone because of the interaction of the computing device with the environment. In this paper we investigate the resilience of Grover's algorithm in the presence of quantum noise. We use the language of density matrices and quantum channels. Our goal is to find the maximal amount of noise, for which the quantum algorithm is better, in terms of mean number of operations, than classical algorithm. We aim to achieve this objective by considering some classes of quantum channels modelling environmentally induced noise.

The paper is organised as follows: in Section II we make a short review of the subject. In Section III we describe the formalism of quantum information theory. In Section IV we present the quantum search algorithm. In Section V we introduce the noise model we have applied to the system. In Section VI we analyse the results and finally in Section VII we present some conclusions.

II. REVIEW OF EXISTING WORK

The problem of influence of noise on the quantum search algorithm has been extensively studied by various researchers. In [5] the authors discuss the influence of classical field upon a quantum system implementing Grover's algorithm. The authors of the paper [6] ask similar question to the one asked in this work, but use Gaussian noise model, which in their case is not

described in the language of quantum channels. In [7] the authors analyse how imperfections in realizations of quantum gates influence the probability of success of the quantum search algorithm. In [8] the authors analyse the behaviour of the quantum search algorithm realized with the use of noisy $\pi/4$ rotation gates.

The effect of unitary noise on the quantum search algorithm is studied in [9]. In [10] the authors examine the robustness of Grover's search algorithm to a random phase error in the oracle and analyse the complexity of the search process. In [11] the author studies decoherence in Grover's quantum search algorithm using a perturbative method. The authors of [12] use the methods of quantum trajectories to study the effects of dissipative decoherence on the accuracy of the Grover's quantum search algorithm. In [13] the author numerically simulates Grover's algorithm introducing random errors of two types: one- and two-qubit gate errors and memory errors.

III. FORMALISM OF QUANTUM INFORMATION

A. Dirac notation

Throughout this paper we use Dirac notation. Symbol $|\psi\rangle$ denotes a complex column vector, $\langle\psi|$ denotes the row vector dual to $|\psi\rangle$. The scalar product of vectors $|\psi\rangle$, $|\phi\rangle$ is denoted by $\langle\psi|\phi\rangle$. The outer product of these vectors is denoted as $|\phi\rangle\langle\psi|$. Vectors are labelled in the natural way: $|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Notation like $|\phi\psi\rangle$ denotes the tensor product of vectors and is equivalent to $|\phi\rangle \otimes |\psi\rangle$.

B. Density operators

The most general state of a quantum system is described by a density operator. In quantum mechanics a density operator ρ is defined as hermitian ($\rho = \rho^\dagger$) positive semi-definite ($\rho \geq 0$) trace one ($\text{tr}(\rho) = 1$) operator. When a basis is fixed the density operator can be written

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in the form of a matrix. Diagonal density matrices can be identified with probability distributions, therefore this formalism is a natural extension of probability theory.

Density operators are usually called quantum states. The set of quantum states is convex [14] and its boundary consists of pure states which in matrix terms are rank one projectors. Convex combinations of pure states lie inside the set and are called mixed states.

1. Entanglement

Entanglement is one of the most important phenomena in quantum information theory. We say that state ρ is separable iff it can be written in the following form

$$\rho = \sum_{i=1}^M q_i \rho_i^A \otimes \rho_i^B, \quad (1)$$

where $q_i > 0$ and $\sum_{i=1}^M q_i = 1$. A state that is not separable is called entangled. It is an open problem of great importance and under investigation, to decide if a given quantum state is entangled or not.

2. Subsystems

Given two states ρ^A, ρ^B of two systems A and B , the product state ρ^{AB} of the composed system is obtained by taking the Kronecker product of the states i.e. $\rho^{AB} = \rho^A \otimes \rho^B$.

Let $[\rho^{AB}]_{kl}$ be a matrix representing a quantum system composed of two subsystems of dimensions M and N . We want to index the matrix elements of ρ using two double indices $[\rho^{AB}]_{m\mu}^{n\nu}$, so that Latin indices correspond to the system A and Greek indices correspond to the system B . The relation between indices is as follows $k = (m-1)N + \mu$, $l = (n-1)N + \nu$. The partial trace with respect to system B reads $\text{tr}_B(\rho^{AB}) = \sum_{\mu} \rho_{n\mu}^{m\mu} = \rho^A$, and the partial trace with respect to system A reads $\text{tr}_A(\rho^{AB}) = \sum_m \rho_{m\nu}^{m\nu} = \rho^B$.

Given the state of the composed system ρ^{AB} the state of subsystems can be found by the means of taking partial trace of ρ^{AB} with respect to one of the subsystems. It should be noted that tracing-out is not a reversible operation, so in a general case

$$\rho^{AB} \neq \text{tr}_A(\rho^{AB}) \otimes \text{tr}_B(\rho^{AB}). \quad (2)$$

C. Completely positive trace-preserving maps (CPTP)

We say that an operation is physical if it transforms density operators into density operators. Additionally we assume that physical operations are linear. Therefore an operation $\Phi(\cdot)$ to be physical has to fulfil the following set of conditions:

1. For any operator ρ its image under operation Φ has to have its trace and positivity preserved i.e. if $\text{tr}(\rho) = 1, \rho \geq 0, \rho = \rho^\dagger$ then $\text{tr}(\Phi(\rho)) = 1, \Phi(\rho) \geq 0, \Phi(\rho) = \Phi(\rho)^\dagger$.
2. Operator Φ has to be linear:

$$\Phi\left(\sum_i p_i \rho_i\right) = \sum_i p_i \Phi(\rho_i). \quad (3)$$

3. The extension of the operator Φ to any larger dimension that acts trivially on the extended system has to preserve positivity. This feature is called complete positivity. It means that for all positive semi-definite $\rho, \xi \geq 0$ the following holds

$$(\Phi \otimes \mathbb{I}_{\dim(\xi)})(\rho \otimes \xi) = \Phi(\rho) \otimes \xi \geq 0. \quad (4)$$

CPTP maps are often called quantum channels.

1. Kraus form

Any operator Φ that is completely positive and trace preserving can be expressed in so called Kraus form [14], which consists of the finite set $\{E_k\}$ of Kraus operators – matrices that fulfil the completeness relation: $\sum_k E_k^\dagger E_k = \mathbb{I}$. The image of state ρ under the map Φ is given by

$$\Phi(\rho) = \sum_k E_k \rho E_k^\dagger. \quad (5)$$

D. Measurement

Quantum states cannot be observed directly. In the literature one considers two main types of measurements: Von Neumann measurement and POVM (Positive Operator Valued Measure) measurement. In this paper we use only Von Neumann measurement but for the sake of completeness we also define POVM measurement.

The mathematical formulation of Von Neumann measurement is given by a map from a set of projection operators to real numbers.

Let us consider an orthogonal complete set of projection operators $P = \{P_i\}_{i=1}^N$ and the set of real measurement outcomes $O = \{o_i\}_{i=1}^N$. Mapping $P \rightarrow O$ is called Von Neumann measurement. Assuming the system is in the state ρ , the probability p_i of measuring outcome o_i is given by the relation $p_i = \text{tr}(P_i \rho)$.

POVM measurement can be considered as a generalisation of Von Neumann measurement. Let us take a set of positive operators $F = \{F_i\}_{i=1}^N$ such that $\sum_{i=1}^N F_i = \mathbb{I}$ and the set of real measurement outcomes $O = \{o_i\}_{i=1}^N$. Mapping $F \rightarrow O$ is called POVM measurement. Given the system is in the state ρ , the probability p_i of measuring outcome o_i is given by the relation $p_i = \text{tr}(F_i \rho)$.

IV. OVERVIEW OF THE GROVER'S ALGORITHM

Grover's unordered database search algorithm is one of the most important quantum algorithms. This is due to the fact that many algorithmic problems can be reduced to exhaustive search.

The main idea of the algorithm is to amplify the probability of the state which represents the sought element. The algorithm is probabilistic and may fail to return the proper result. Fortunately the probability of success is reasonably high.

A. The problem

Let X be a set and let $f : X \rightarrow \{0, 1\}$, such that

$$f(x) = \begin{cases} 1 \Leftrightarrow x = x_0 \\ 0 \Leftrightarrow x \neq x_0 \end{cases}, x \in X, \quad (6)$$

for some marked $x_0 \in X$.

For the sake of simplicity we assume that X is a set of binary strings of length n . Therefore $|X| = 2^n$ and $f : \{0, 1\}^n \rightarrow \{0, 1\}$. We can map the set X to a set of states over $\mathbb{C}^{\otimes 2^n}$ in the natural way: $x \leftrightarrow |x\rangle$, forming orthogonal, complete set of vectors. The goal of the algorithm is to find the marked element.

B. The algorithm

The Grover's algorithm is composed of two main procedures: the oracle and diffusion.

1. Oracle

By an oracle we call a function that marks one defined element. In the case of Grover's algorithm, the marking of the element is done by the negation of the amplitude of the sought state.

With the use of elementary quantum gates the oracle can be constructed using ancilla $|q\rangle$ in the following way:

$$O|x\rangle|q\rangle = |x\rangle|q \oplus f(x)\rangle, \quad (7)$$

where \oplus denotes addition modulo 2. If the register $|q\rangle$ is prepared in the state

$$|q\rangle = H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}, \quad (8)$$

where H denotes the Hadamard gate, then, by substitution, Eq. (7) can be written as

$$O|x\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}} = (-1)^{f(x)}|x\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}}. \quad (9)$$

By tracing out the ancilla we get

$$O|x\rangle = -(-1)^{f(x)}|x\rangle. \quad (10)$$

2. Diffusion

The operator D rotates any state around the state

$$|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle, \quad (11)$$

where D can be written as

$$D = -H^{\otimes n}(2|0\rangle\langle 0| - \mathbb{I})H^{\otimes n} = 2|\psi\rangle\langle\psi| - \mathbb{I}. \quad (12)$$

3. Initialisation

We begin in the ground state $|0 \dots 0\rangle$. In the first step of the algorithm we apply the Hadamard gate $H^{\otimes n}$ on the entire register. This transforms the initial state into flat superposition of computational base states:

$$H^{\otimes n}|0 \dots 0\rangle = \frac{1}{\sqrt{2^n}} (|0 \dots 0\rangle + \dots + |1 \dots 1\rangle). \quad (13)$$

4. Grover iteration

The core of the algorithm consists of the applications of so called Grover iteration gate $G = D \cdot O$. This procedure causes the sought state to be amplified and others states to be attenuated.

5. Number of iterations

The application of the diffusion operator on the base state $|x\rangle$ gives

$$D|x\rangle = -|x\rangle + \frac{2}{N} \sum_y |y\rangle. \quad (14)$$

The application of this operator on any state gives

$$\begin{aligned} D|x\rangle &= \sum_i \alpha_i (-|x\rangle + \frac{2}{N} \sum_y |y\rangle) \\ &= \sum_i (-\alpha_i + 2s) |x\rangle, \end{aligned}$$

where

$$s = \frac{1}{N} \sum_i \alpha_i. \quad (15)$$

k -fold application of Grover's iteration G on initial state $|s\rangle$ leads to [4, 15]

$$G^k|s\rangle = \alpha_k \sum_{x \neq x_0} |x\rangle + \beta_k |x_0\rangle, \quad (16)$$

with real coefficients:

$$\alpha_k = \frac{1}{\sqrt{N-1}} \cos(2k+1)\theta, \quad \beta_k = \sin(2k+1)\theta, \quad (17)$$

where θ is an angle that fulfils the relation

$$\sin(\theta) = \frac{1}{\sqrt{N}}. \quad (18)$$

Therefore the coefficients α_k, β_k are periodic functions of k . After the series of iterations β_k rises. The influence of the marked state $|x_0\rangle$ on the state of the register results in the evolution of the initial state $|s\rangle$ towards the marked state.

The β_k attains its maximum after approximately $\frac{\pi}{4}\sqrt{N}$ steps. The number of steps needed to transfer the initial state towards the marked state is of order $O(\sqrt{N})$. In the classical case the number of steps is of order $O(N)$.

6. Measurement

The last step of the Grover's algorithm is Von Neumann measurement. The probability of obtaining the proper result is $|\beta_k|^2$.

V. NOISE MODEL

The above discussion of quantum search algorithm has been conducted using state vector formalism. In order to incorporate the noise into the quantum computation model we need to make use of density operators which define the quantum state in the most general way.

A. Quantum noise

Microscopic systems that are governed by the laws of quantum mechanics are hard to control and at the same time, to separate from the environment. The interaction with the environment introduces noise into the quantum system. Therefore any future quantum computer will also be prone to noise.

a. One-qubit noise There are several one-parameter families of one-qubit noisy channels that are typically discussed in the literature [16]. We present them briefly below.

Depolarising channel This is a bi-stochastic channel that transforms any state into maximally mixed state with a given probability α . The family of channels can be defined using a four-element set of Kraus operators

$$\left\{ \sqrt{1-\alpha}\mathbb{I}, \sqrt{\frac{\alpha}{3}}\sigma_x, \sqrt{\frac{\alpha}{3}}\sigma_y, \sqrt{\frac{\alpha}{3}}\sigma_z \right\},$$

where

$$\mathbb{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

are Pauli matrices.

Amplitude damping The amplitude damping channel transforms $|1\rangle$ into $|0\rangle$ with a given probability α . State $|0\rangle$ remains unchanged. The set of Kraus operators is following

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\alpha} \end{bmatrix}, \begin{bmatrix} 0 & \sqrt{\alpha} \\ 0 & 0 \end{bmatrix} \right\}.$$

Phase damping Phase damping is purely quantum phenomenon which describes the loss of quantum information without the loss of energy. It is described by the following set of Kraus operators

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\alpha} \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{\alpha} \end{bmatrix} \right\}.$$

Bit flip The bit flip family of channels is the quantum version of classical Binary Symmetric Channel. The action of the channel might be interpreted in the following way: it flips the state of a qubit from $|0\rangle$ to $|1\rangle$ and from $|1\rangle$ to $|0\rangle$ with probability α . Kraus operators for this family of channels consist of a matrix proportional to the identity and a matrix proportional to the negation gate

$$\{\sqrt{1-\alpha}\mathbb{I}, \sqrt{\alpha}\sigma_x\}.$$

Phase flip The phase flip channel acts similarly to bit flip channel with the distinction that σ_z gate is applied randomly to the qubit

$$\{\sqrt{1-\alpha}\mathbb{I}, \sqrt{\alpha}\sigma_z\}.$$

Bit-phase flip The bit-phase flip channel may be considered as joint application of bit and phase flip gates on a qubit. Its Kraus operators form is as follows

$$\{\sqrt{1-\alpha}\mathbb{I}, \sqrt{\alpha}\sigma_y\}.$$

In all the above families of channels the real parameter $\alpha \in [0, 1]$ can be interpreted as the amount of noise introduced by the channel.

b. Multiqubit local channels Our goal is to extend the noise acting on distinct qubits to the entire registers. We assume that the appearance of an error on a given qubit is independent from an error appearing on any other qubits.

In order to apply noise operators to multiple qubits we form a new set of Kraus operators acting on a larger Hilbert space.

We assume that we have the set of n one-qubit Kraus operators $\{e_k\}_{k=1}^n$. We construct the new set of n^N operators $\{E_k\}_{k=1}^{n^N}$ that act on Hilbert space of dimension

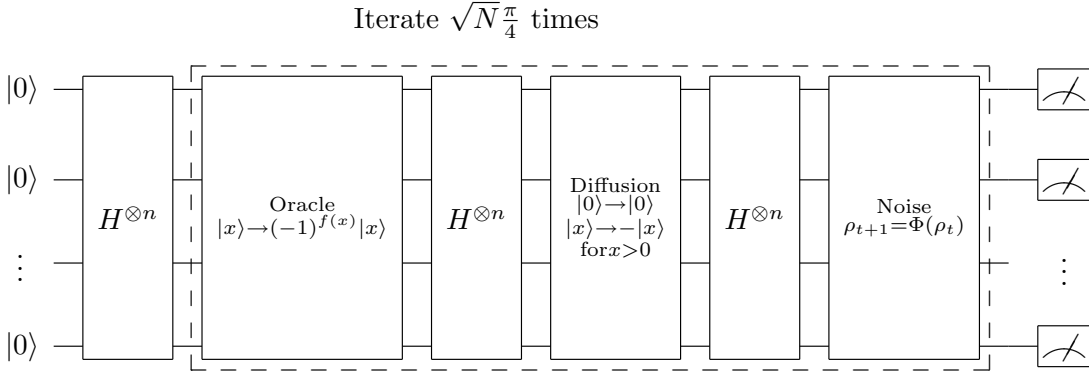


FIG. 1. The circuit for Grover's algorithm extended with a non-unitary noisy channel.

2^N by applying the following formula

$$\{E_k\} = \bigcup_I \{e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_N}\}, \quad (19)$$

where $I = \{i_1\}_{i_1=1}^n \times \{i_2\}_{i_2=1}^n \times \dots \times \{i_N\}_{i_N=1}^n$.

One should note that the extended channel $\Phi(\rho) = \sum_k E_k \rho E_k^\dagger$ is by the definition local [14].

By applying Eq. (19) to the sets of operators listed above we obtain one-parameter families of local noisy channels, which we use in further investigations.

B. Application of noise to the algorithm

In order to simulate noisy behaviour of the system implementing the algorithm we apply a noisy channel after every Grover iteration. The evolution of the system is described by the following procedure, which is graphically depicted in Fig. 1

1. Prepare system in state $\rho_0 := |0^{\otimes n}\rangle\langle 0^{\otimes n}|$.
2. $\rho := H^{\otimes n} \rho_0 H^{\otimes n\dagger}$
3. $\lfloor \frac{\pi}{4} \sqrt{N} \rfloor$ times do:
 - (a) apply Grover iteration $\rho := G \rho G^\dagger$,
 - (b) apply noise $\rho := \Phi(\rho)$.
4. Perform orthogonal measurement in computational basis. The probability of finding the sought element ξ is $p = \langle \xi | \rho | \xi \rangle$.

This approach simplifies the physical reality but it is sufficient to study the robustness of the algorithm in the presence of noise. In order to study the discussed problem we make use of the numerical simulation. Therefore some simplification is necessary as the size of the problem grows exponentially fast with the number of qubits.

The tool we use is **quantum-octave** [17], a library that contains functions for simulation and analysis of quantum processes.

In our model we assume that it is easy to verify if the correctness of the quantum search algorithm the result of quantum search results. It is an assumption usually made in the complexity analysis of search algorithms.

VI. ANALYSIS OF THE INFLUENCE OF NOISE ON THE EFFICIENCY OF THE ALGORITHM

An interesting question arises: “What is the maximal amount of noise for which Grover's algorithm is more efficient than any classical search algorithm?”

Grover's algorithm is probabilistic, therefore we cannot expect to obtain a valid outcome with certainty. We assume that if algorithm fails in a given run we will re-run it. There is a certain number of reruns for which quantum algorithm is worse than classical. We are interested only in the statistical behaviour of algorithm and calculate the mean value of repetitions.

Let $k = \lfloor \frac{N}{2} / \frac{\pi}{4} \sqrt{N} \rfloor$ be the maximal number of single runs of Grover's algorithm for which quantum searching is faster than classical.

We compute p_{\min} minimal value of success probability of single run of Grover's algorithm for which we obtain a valid result with confidence C

$$p_{\min} = \min_p \{1 - (1 - p)^k \geq C\}. \quad (20)$$

Numerically obtained values of p_{\min} for confidence level $C = 0.95$ for Grover's algorithm are listed in Tab. I.

For our numerical experiment we assume that sought element ξ lies in the “middle” of the space of elements *i.e.* $\xi = 2^{n-1}$.

Plots in Fig. 2 depict the influence of noise parameter α on a successful run of Grover's algorithm acting on six qubits. These values of parameter α for which the plots are above threshold level p_{\min} can be considered as the amounts of noise which do not make quantum search algorithm less efficient than classical search algorithms.

We can compare the probabilities from plots in Fig. 2 and these for other sizes of quantum registers with p_{\min}

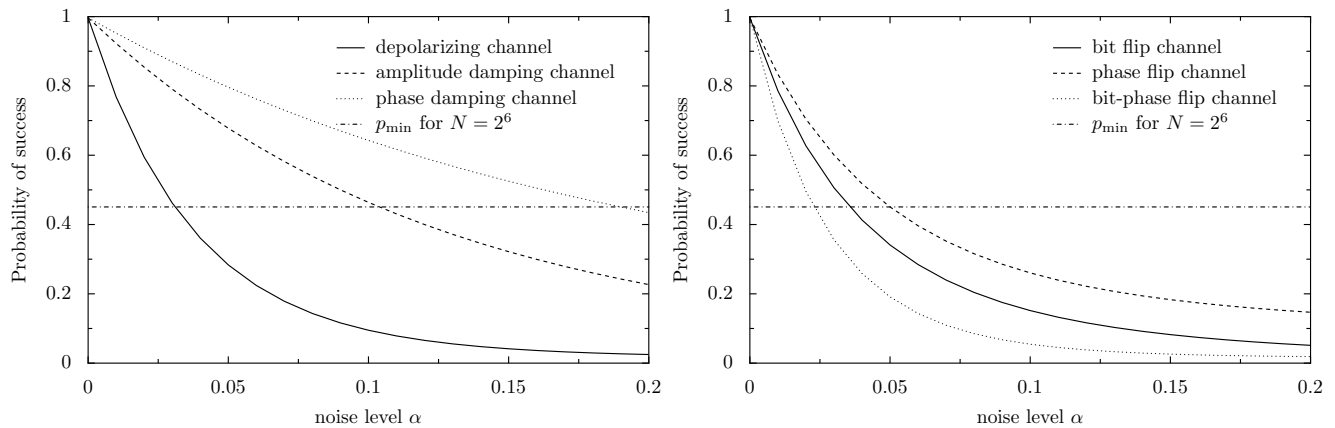


FIG. 2. Probabilities of successful run of Grover's algorithm in function of the noise parameter α . Case of six qubits. The value for which the plots attain p_{\min} threshold is shown in Tab. I.

| Size of the system | k | p_{\min} |
|--------------------|-----|------------|
| $N = 2^3$ | 1 | 0.95000 |
| $N = 2^4$ | 2 | 0.77639 |
| $N = 2^5$ | 3 | 0.63160 |
| $N = 2^6$ | 5 | 0.45072 |
| $N = 2^7$ | 7 | 0.34816 |
| $N = 2^8$ | 10 | 0.25887 |

TABLE I. Values of k and p_{\min} for Grover's algorithm.

and find the value of the noise parameter α for which it is equal to p_{\min} . The results of the comparison are collected in Tab. II for confidence level $C = 0.95$ and for the channels we have described in Section V.

| $C = 0.95$ | depolarising | amplitude damping | phase damping |
|------------|--------------|-------------------|---------------|
| $N = 2^4$ | 0.025 | 0.069 | 0.177 |
| $N = 2^5$ | 0.032 | 0.010 | 0.204 |
| $N = 2^6$ | 0.031 | 0.104 | 0.190 |
| $N = 2^7$ | 0.026 | 0.094 | 0.158 |
| $N = 2^8$ | 0.020 | 0.075 | 0.122 |

| | bit flip | phase flip | bit-phase flip |
|-----------|----------|------------|----------------|
| $N = 2^4$ | 0.025 | 0.047 | 0.018 |
| $N = 2^5$ | 0.032 | 0.054 | 0.024 |
| $N = 2^6$ | 0.031 | 0.050 | 0.023 |
| $N = 2^7$ | 0.026 | 0.041 | 0.020 |
| $N = 2^8$ | 0.020 | 0.031 | 0.015 |

TABLE II. The maximal values of noise parameter α for which Grover's search algorithm is as efficient as classical search algorithm in terms of number of uses of the oracle.

In the case of three qubits we have found that, if we expect confidence level $C = 0.95$ or higher, Grover's algorithm is never better than classical search algorithm. It means that if we want to get the result with high probability we need to repeat the quantum search so many

times that it is more efficient to perform this task classically.

In other cases we have obtained the values of the noise parameter α between ~ 0.010 and ~ 0.2 depending on the noise type and the size of the system. We observe that even if the amount of noise is larger in bigger systems (what causes the algorithm to be less efficient) the noise is compensated by the quantum speed-up.

The results gathered in Tab. II do not form a monotonic pattern. To understand this fact we have to take into account that two factors influence these numbers. The first one is due to the fact that the same value of noise parameter α has larger influence on the quantum system for bigger numbers of qubits and for larger N the number of Grover iterations and noisy channel applications k raises. At the same time the more qubits are used to perform the search algorithm the more important the quantum speed-up is.

VII. SUMMARY

In this work we have shown that a new way of analysing the influence of quantum noise on the quantum search algorithm. Our method uses the model of density matrices and quantum channels represented in Kraus form.

We can conclude that the simulations and analysis have shown that only for small amounts of noise the quantum search algorithm is still more efficient than any classical algorithm.

From our numerical results we conclude that different forms of noise have different impact on the efficiency of the quantum search algorithm. The least destructive form of noise is phase damping, more destructive is amplitude damping and the most destructive is the depolarizing channel.

Further work would have to take into account quantum error correcting codes and more precise noise models dependent on the implementation. One of the research

directions would be to analyse the quantum search algorithm in the framework of control Hamiltonians taking into account Markovian approximation of quantum noise.

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